Transient Response of an Infinite Cylindrical Antenna

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Abstract—**A simple analytical formula derived from Wu's exact solution for the transient response of an infinite cylindrical antenna is found to be extremely accurate. Existing results from other numerical schemes are also compared to Wu's solution and found to be in good agreement.**

I. INTRODUCTION

Wu [1], [2] and Brundell [3] independently provided an exact solution of the transient response of an infinite cylindrical antenna by a Fourier transform technique. This result was rederived by Morgan [4], and by Lee and Latham [5] by different procedures of the contour integration. This result is quite useful for providing checks on the numerous numerical schemes. Unfortunately, these important checks have never been done in the past. Therefore, the first objective of this communication is to provide accuracy checks on existing numerical results obtained by typical time-domain schemes [6], [7].

The second objective here is to show that, although the exact solution is in terms of a complicated one-dimensional integral, a simple analytical formula can be derived that describes the overall waveform.

II. TRANSIENT SOLUTION

The transient current of an infinite cylindrical antenna in free space driven at the delta gap by a unit step voltage is [1], [2]

$$\mathcal{J}(z,t) = \frac{4V}{\pi\zeta_0}\int_0^\infty \frac{J_0(\zeta\tau)\,d\zeta}{\{[J_0(\zeta)]^2 + [Y_0(\zeta)]^2\}\zeta}. \tag{1}$$

$$\tau = \frac{\sqrt{c^2t^2 - z^2}}{a} \tag{2}$$

ζ_0 = free space impedance.

Note that the transient antenna current is only a function of τ. The physical interpretation of τ is the ratio of the transverse-wavefront-to-wire distance to the wire radius (see Fig. 1). An analogy to the coaxial line solution but with expanding outer radius can be drawn. Of course, (1) describes only the exterior current. The interior current unique to a tubular antenna is not discussed here. Since (1) is not a convenient form of numerical integration, Wu [1] derived for that purpose an alternate formula:

$$\mathcal{J}(z,t) = \frac{2\pi V}{\zeta_0}\int_0^\infty \frac{d\zeta}{\zeta}\,\frac{K_0[\zeta\tau]I_0(\zeta)}{K_0(\zeta)\{[K_0(\zeta)]^2 + \pi^2[I_0(\zeta)]^2\}}. \tag{3}$$

In using either (1) or (3) for numerical integration, it is important to observe that the contribution near $\zeta = 0$ to the integral is quite significant and requires a separate analytical treatment. The curve labeled as Wu's in Fig. 2 was produced by numerical integration of form (3) with essentially the same result obtained from (1). The numerical accuracy of these calculations is discussed below. In that figure the driving point current for a step voltage excitation is given in units of milliamps. The other two curves were obtained by numerical zoning of the time-domain integral equation [6], [7]. These two curves are from the early-time response at the driving point of a center-driven finite cylindrical antenna of length $2L$ with $L/a = 74.2$ and $L/c = 1.67$ ns. As can be seen from Fig. 2, the agreement is quite good except for the very early time.

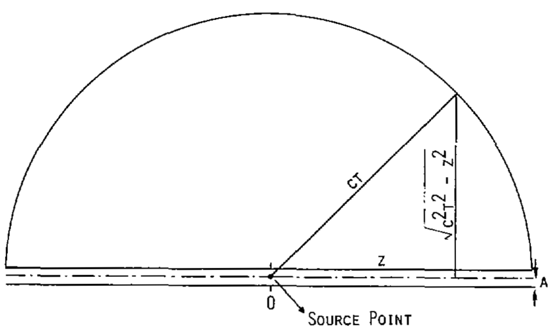

Fig. 1. Interpretation of the transverse-wavefront-to-wire distance. Wavefront is spherical with center at the source. Transverse-wavefront-to-wire distance is $\sqrt{c^2t^2 - z^2}$.

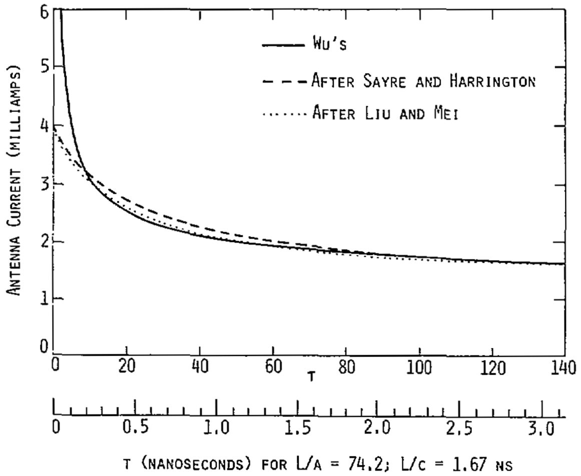

Fig. 2. Driving point current ($z = 0$) for a center fed linear antenna with a unit step voltage excitation.

III. ASYMPTOTIC EVALUATION OF (1) FOR LARGE τ

In this section, the main result, (7), is obtained. Although a detailed derivation is not given, the reader should note that (7) not only provides numerically accurate results, but also gives rise to the known time harmonic formula.

Here we apply a method developed by Wu [2], [8] for the time harmonic response of a long antenna to evaluate (1). It is important to realize that when τ is large, the major contribution to the integration comes from small ζ. However, we can not assume the argument $\zeta\tau$ in $J_0(\zeta\tau)$ as large or small. This is the most important feature in our asymptotic evaluation. Approximating $J_0(\zeta) \sim 1$ and $Y_0(\zeta) \sim 2/\pi \ln(\zeta/2) + \gamma)$, we write (1) as

$$\mathcal{J}(z,t) \sim \frac{\pi V}{\zeta_0}\int_0^\infty \frac{d\zeta}{\zeta}\,\frac{J_0(\zeta\tau)}{\dfrac{\pi^2}{4} + \left(\ln\left(\dfrac{\zeta}{2}\right) + \gamma\right)^2}. \tag{4}$$

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Integrating (4) by parts leads to

$$J(z,t) \sim \frac{2V\tau}{\zeta_0}\int_0^\infty J_1(\zeta\tau)\arctan\left[\frac{\pi}{-2\left(\ln\left(\frac{\zeta}{2}\right)+\gamma\right)}\right]d\zeta \tag{5}$$

since arctan $[\pi/-2\,(\ln(\zeta/2)+\gamma)]$ is a slowly varying function of ζ, it is possible to replace $\ln(\zeta)$ by the average

$$\left[\int_0^\infty d\zeta J_1(\zeta\tau)\right]^{-1}\int_0^\infty J_1(\zeta\tau)\ln(\zeta)\,d\zeta = -\left(\ln\left(\frac{\tau}{2}\right)+\gamma\right) \tag{6}$$

Equation (6) is obtained from Gradshteyn *et al.* [9].

Use of (6) in (5) gives

$$\begin{aligned} J(z,t) &\sim \frac{2V}{\zeta_0}\arctan\left(\frac{\pi}{2\ln(\tau)}\right)\\ &= \frac{V}{\zeta_0 i}\ln\left(\frac{\ln(\tau)+\frac{i\pi}{2}}{\ln(\tau)-\frac{i\pi}{2}}\right). \end{aligned} \tag{7}$$

Equation (7) can be improved upon by accounting for the variation of $\ln\zeta$ from its average. However, since (7) is already an excellent approximation, this correction is not given here. Let us expand (7) in a small argument approximation to give

$$J(z,t) \sim \frac{V\pi}{\zeta_0\ln(\tau)}. \tag{8}$$

Using another expansion technique, a similar but different formula was derived by Lee and Latham [5]. Although (8) can be interpreted in a transmission line analogy, it should be pointed out that this antenna response bears no resemblance to the lossless or lossy transmission line solution.

For comparison purposes both (1) and (3) were numerically integrated. An analytical approximation was made to both integrals by taking the limit as $\zeta \to 0$. The remaining integrals (0+) were then computed piecewise over adjoining finite limits. Each piece was integrated using an adaptive order 40 Gaussian quadrature with a required relative error convergence criterion of 10^{-8}. Although the rate of convergence of (1) is a function of τ, at least three decimal place accuracy was achieved for τ values in Table I. Equation (3) was computed to within $\pm 10^{-5}$ accuracy for the same τ values.

The agreement of these formulas with the numerical integration is shown in Fig. 3. In that figure, (7) is represented by the dashed line, (8) is represented by the dotted line, and the numerical integration (Wu's) is illustrated by the solid line. A tabulated comparison for selected values of τ is given in Table I. Equation (7) is found to be quite accurate even for $\tau = 1$, at which the discrepancy is only about 7 percent.

Since the current response can be expressed in a very simple formula, (7), it is instructive to examine the surrounding fields. To do this we write the θ-component of $\vec{B}$ around the wire [2], [5]:

$$B_\theta = \frac{\mu_0 V}{\pi\zeta_0 a}\int_0^\infty \frac{J_1\left(\frac{\rho}{a}\zeta\right)Y_0(\zeta)-J_0(\zeta)Y_1\left(\frac{\rho}{a}\zeta\right)}{J_0^2(\zeta)+Y_0^2(\zeta)}J_0(\zeta\tau)\,d\zeta. \tag{9}$$

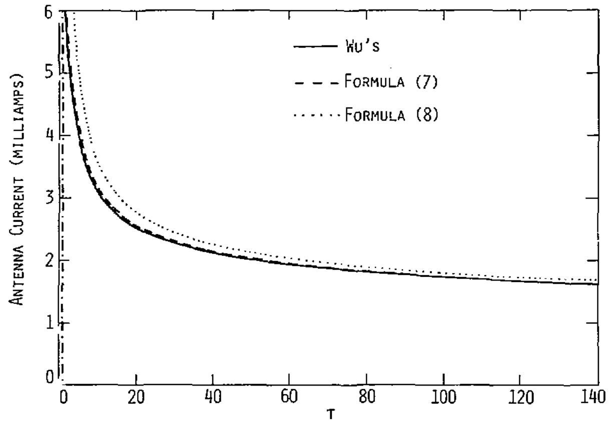

Fig. 3. Comparison of Wu's formula (1) or (3) (solid line), approximate formulas (7) (dashed line), and (8) (dotted line).

Equation (9) can be evaluated in the same manner. The resulting approximation is

$$B_\theta \sim \frac{\mu_0 J}{2\pi\rho}, \qquad \text{for } \frac{\rho}{a} \ll \tau \tag{10}$$

with J given by (7).

To obtain a complete solution for B_θ, E_z, and E_ρ the uniform asymptotic expansions of the relevant integrals such as (9) for ρ/a and τ would be needed. However, this task is not pursued here.

IV. TIME HARMONIC FORMULA

Finally, in order to show that (7) gives rise to the known existing formula in the time harmonic case [2], [8], consider the Fourier transformation of (7):

$$\begin{aligned} I(z,\omega) &= \int_0^\infty J(z,t)e^{i\omega t}\,dt\\ &\sim \frac{V}{\zeta_0 i}\int_0^\infty \ln\left(\frac{\ln\sqrt{c^2t^2-z^2}-\ln(a)+i\frac{\pi}{2}}{\ln\sqrt{c^2t^2-z^2}-\ln(a)-i\frac{\pi}{2}}\right)\\ &\quad\cdot e^{i\omega t}\,dt. \end{aligned} \tag{11}$$

Let $\tau_1 = ct - z$, and $\eta_1 = -i\tau_1$, then (11) can be reduced to

$$\begin{aligned} I(z,\omega) &= \frac{Ve^{ikz}}{\zeta_0 c}\int_0^\infty \ln\left(\frac{\ln(\eta_1)+\ln(2z)-2\ln(a)+i\frac{3\pi}{2}}{\ln(\eta_1)+\ln(2z)-2\ln(a)-i\frac{\pi}{2}}\right)\\ &\quad\cdot e^{-\eta_1\frac{\omega}{c}}\,d\eta_1. \end{aligned} \tag{12}$$

For large ω/c, since the contribution to (11) comes mostly from $t \sim 0$, the following approximation is used in arriving at (12):

$$ct + z = 2z + i\eta_1 \sim 2z.$$

Equation (12) is approximated in the same way as (5). Here the averaging required is

$$\int_0^\infty \ln(\eta_1)e^{-k\eta_1}\,d\eta_1 = \frac{1}{k}\,[-\ln(k)-\gamma] \tag{13}$$

TABLE I
COMPARISON OF APPROXIMATE FORMULAS TO WU'S EXACT SOLUTION

τ	Numerical Evaluation of Wu's Exact Solution	Formula 7	Formula 8
1.00	9.001	8.333	∞
1.25	7.855	7.590	37.371
1.50	7.073	6.998	20.567
1.75	6.501	6.522	14.901
2.00	6.061	6.133	12.031
2.25	5.711	5.809	10.283
2.50	5.424	5.536	9.101
2.75	5.184	5.302	8.243
3.00	4.980	5.099	7.591
3.50	4.649	4.765	6.657
4.00	4.391	4.500	6.015
5.00	4.010	4.105	5.181
7.50	3.448	3.515	4.139
10.00	3.127	3.178	3.622
12.00	2.949	2.993	3.356
15.00	2.756	2.790	3.079
20.00	2.538	2.564	2.784
30.00	2.280	2.297	2.452
50.00	2.017	2.027	2.132
75.00	1.846	1.852	1.931
100.00	1.741	1.745	1.811
150.00	1.610	1.613	1.664
250.00	1.471	1.471	1.510
500.00	1.315	1.314	1.342
1000.00	1.188	1.187	1.207

The resulting formula for $I(z, \omega)$ is

$$I(z,\omega) \sim -\frac{Ve^{ikz}}{\zeta_0\omega}\ln\left(1-\frac{i2\pi}{\ln(2kz)-2\ln(ka)-\gamma+i\dfrac{3\pi}{2}}\right)$$

$$=\frac{iVe^{ikz}}{(-i\omega)\zeta_0}\ln\left(1-\frac{i2\pi}{\ln(2kz)-2\ln(ka)-\gamma+i\dfrac{3\pi}{2}}\right). \tag{14}$$

This agrees with that given in the literature [10] except for the extra factor $1/(-i\omega)$. The difference here is the use of excitations: delta function versus unit step function.

V. CONCLUSION

Wu's exact solution for transient response of an infinite cylindrical antenna is compared to results from approximate numerican techniques in Fig. 1 and is found to be in good agreement. The exact solution is asymptotically evaluated to give an extremely simple and amazingly accurate formula. Comparison of this formula and the numerical result is given in Fig. 2 and Table I.

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Current Induced by a Plane Wave on a Thin Infinite Coated Wire Above the Ground

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Abstract—**The current induced on a thin infinite coated wire above the ground when an electromagnetic plane wave is incident on it is calculated. The final result is presented in a form which exhibits clearly the various contributions to the total impedance of the wire-ground system. Numerical examples are given which demonstrate the dependence of the induced current on the conductivity of the ground, the thickness of the coating, and the polarization and direction of propagation of the incident wave.**

I. INTRODUCTION

The propagation and the excitation of electromagnetic waves on long cables parallel to an interface between two homogeneous half-spaces has been the subject of a number of recent works [1]-[7]. However, in the discussions of an infinite wire in the air, only the bare wire case seems to have been treated exactly [4], [5]. In the present work we calculate the current which is induced on an infinite coated wire above the ground by an incident plane wave. We employ the method developed by Wait [2] by which he has treated in detail the case of a buried insulated wire [3], [7].

II. FORMULATION

We consider an infinitely long thin wire of radius r_w and conductivity σ_w, located at a height h over the ground. The wire is

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